Modulational instability in optical-microwave interaction

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The stability of continuous optical and microwave fields is studied in the presence of dispersion and second order nonlinearity. The cascade combination of optical rectification and the electro-optic effect induces modulational instability (MI) in a wide range of system parameters. It is demonstrated that MI can lead potentially to filamentation of high power optical pulses as well as the generation of terahertz radiation.

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I. INTRODUCTION

Continuous waves (CW) with amplitudes independent of spatial or temporal variables are simple stationary solutions of a variety of nonlinear wave equations. The interplay between dispersion and nonlinearity can lead to an exponential growth of small perturbations and finally to the break up of the CW solution, a phenomenon usually referred to as modulational instability (MI).

MI of optical waves is well known to occur in media with cubic nonlinearities, where wave propagation is governed by the nonlinear Schrödinger equation and its derivatives [1]. However, it was shown recently that cascading effects between a fundamental and its second harmonic (SHG cascading) in a system with quadratic nonlinearity can also lead to an instability of stationary CW solutions [2,3].

Whereas SHG cascading is now well understood (see Ref. [4] and references therein), a less investigated effect present in media with a quadratic nonlinearity is optical rectification (OR), i.e., the generation of a static or quasistatic electric field from an optical input field. The effect of OR was experimentally shown for the first time by Brass *et al.* in 1962 [5]. Gustafson *et al.* [6] suggested the self-modulation of optical pulses by a cascaded process of optical rectification and the linear electro-optic effect. Recently Bosshardt *et al.* [7] demonstrated a dominant contribution of the rectification-induced cubic nonlinearity to the nonlinearly induced refractive index change in $KNbO_3$ for certain crystal orientations.

In this paper we show that the coupling between an optical and a microwave field in a medium with second-order nonlinearity can lead to an instability of the CW solutions. Here we consider evolution equations derived for the interaction of a microwave with an optical wave in a waveguiding structure with lateral confinement, e.g., an electro-optic modulator [8,9]. The resulting system is an example for the nonlinear interaction of long waves with short waves, the latter described by a complex envelope function with an underlying highly oscillating modulation signal [10]. Other examples for the long-wave–short-wave interaction are the coupling of gravity and capillary modes of surface water waves [11] or the copropagation of electron plasma with ion-acoustic waves [12]. Whereas the following discussion should be applicable to those cases, the main emphasis in this paper is laid on the interaction of an optical wave with a microwave. As will be demonstrated, MI can lead to a filamentation of high power optical pulses. Furthermore, as the generated microwave signal develops a similar kind of modulation, MI is shown potentially to result in the generation of narrow-bandwidth terahertz signals.

II. EVOLUTION EQUATIONS AND MODULATIONAL INSTABILITY

Starting points are the scaled scalar interaction equations [8,9]

$$\left(\frac{\partial}{\partial z}+\delta\frac{\partial}{\partial t}-\sigma_m\frac{\partial^3}{\partial t^3}\right)u_m-\frac{\partial}{\partial t}|u_o|^2=0, \tag{1}$$

$$\left(\mathrm{i}\frac{\partial}{\partial z}-\frac{\sigma_o}{2}\frac{\partial^2}{\partial t^2}-u_m\right)u_o=0. \tag{2}$$

Here u_m is the real valued amplitude of the microwave (long wave) and u_o is the complex, slowly varying envelope of the optical wave (short wave). The coordinates z and t correspond to the propagation direction and temporal evolution in a reference frame moving with the velocity of the optical wave. The parameter δ describes the velocity mismatch between both waves. $\sigma_m=\pm 1$ and $\sigma_o=\pm 1$ are the signs of the dispersion coefficients of the microwave signal and the optical wave, respectively. Equations (1) and (2) constitute the most simple, but still accurate way, to describe the interaction between the optical and microwave fields. In particular Eq. (2) solely conserves the optical power. This is well justified for all realistic geometries, where the overall conversion efficiency is small. If considerable amounts of energy are transferred to the microwave domain, further terms have to be added to Eq. (2) [9]. However, the appearance of MI is only weakly influenced by a power exchange between the optical and microwave components.

A simple stationary CW solution of Eqs. (1) and (2) is given by

$$u_o=a_o\exp[\mathrm{i}(\beta z-\omega t)],\qquad u_m=a_m, \tag{3}$$

with the dispersion relation for the optical wave

$$\beta=\frac{\sigma_o}{2}\omega^2-a_m. \tag{4}$$

Solutions (3) and (4) refer to a continuous optical wave with constant amplitude a_o (assumed to be real) that is subject to a phase shift due to the electro-optic effect proportional to a constant dc background field a_m. A solution with a nonzero frequency ω corresponds simply to a change of the carrier frequency of the optical wave. As this has no influence on the general behavior, apart from a change of the velocity mismatch between both waves, only the case $\omega=0$ will be considered.

In the following, we are interested in the stability of the CW solution against small perturbations. We hence introduce small changes $\Delta a_o(z,t)$, $\Delta a_m(z,t)$ from the stationary solution as

$$u_o=[a_o+\Delta a_o(z,t)]\exp[-ia_m z], \tag{5}$$

$$u_m=a_m+\Delta a_m(z,t). \tag{6}$$

Inserting Eqs. (5) and (6) into Eqs. (1) and (2), and keeping only first order terms in $\Delta a_o(z,t)$ and $\Delta a_m(z,t)$, corresponding to small perturbations, one obtains

$$\left(\frac{\partial}{\partial z}+\delta\frac{\partial}{\partial t}-\sigma_m\frac{\partial^3}{\partial t^3}\right)\Delta a_m-a_o\frac{\partial}{\partial t}(\Delta a_o+\Delta a_o^*)=0, \tag{7}$$

$$\left(\mathrm{i}\frac{\partial}{\partial z}-\frac{\sigma_o}{2}\frac{\partial^2}{\partial t^2}\right)\Delta a_o-a_o\Delta a_m=0. \tag{8}$$

The set of equations (7) and (8) does not contain the constant background field a_m indicating that a constant voltage applied to the microwave strip line shifts the optical phase only [see Eq. (4)], but does not influence the microwave. In what follows, we express the perturbations as oscillatory functions:

$$[\Delta a_m,\Delta a_o,\Delta a_o^*]^T=[\epsilon_m(z),\epsilon_o(z),\epsilon_o^*(z)]^T\exp(i\Omega t). \tag{9}$$

Note that Δa_o and Δa_o^* denote small perturbations of the optical wave and its complex conjugate and have to be varied independently. Substituting Eq. (9) into Eqs. (7) and (8) leads to the eigenvalue problem

$$\frac{\partial}{\partial z}[\epsilon]=i\bar{M}[\epsilon], \tag{10}$$

where $[\epsilon]=[\epsilon_m,\epsilon_o,\epsilon_o^*]^T$ and the matrix $\bar{M}$ is given by

$$\bar{M}=\begin{bmatrix}-[\delta\Omega+\sigma_m\Omega^3] & a_o\Omega & a_o\Omega\\ -a_o & \dfrac{\sigma_o\Omega^2}{2} & 0\\ a_o & 0 & -\dfrac{\sigma_o\Omega^2}{2}\end{bmatrix}. \tag{11}$$

Unstable, i.e., growing, frequencies Ω are given by eigenvalues λ of the matrix $\bar{M}$ with negative imaginary part. The eigenvalues are determined by a cubic equation,

$$\lambda^3+c_1\lambda^2-c_2\lambda+c_3-c_1c_2=0, \tag{12}$$

with real valued coefficients $c_1=\Omega(\delta+\sigma_m\Omega^2)$, $c_2=\Omega^4/4$, and $c_3=\sigma_o a_o^2\Omega^3$. The solutions of Eq. (12) are given by

$$\lambda_1=u+v-\frac{c_1}{3}, \tag{13}$$

$$\lambda_{2,3}=-\frac{u+v}{2}-\frac{c_1}{3}\pm i\frac{\sqrt{3}(u-v)}{2}, \tag{14}$$

where,

$$u=\sqrt[3]{-q+\sqrt{D}},\quad v=\frac{p}{u},\quad \text{with}\quad D=q^2-p^3,$$

$$p=\frac{c_2}{3}+\frac{c_1^2}{9},\quad q=\frac{1}{3}c_1\left(\frac{1}{9}c_1^2-c_2\right)+\frac{c_3}{2}. \tag{15}$$

For $D>0$, we obtain a pair of complex eigenvalues λ_2,λ_3 and the system is modulationally unstable. $D=0$ marks hence the boundary to unstable domains. The MI gain for an unstable solution is given by $g=|\mathrm{Im}(\lambda_{2,3})|$. The gain does not depend on the sign of Ω resulting in symmetric sidebands around the carrier frequency of the optical wave. As the MI gain is invariant against the transformations $\sigma_o\to-\sigma_o$, $\sigma_m\to-\sigma_m$, and $\delta\to-\delta$, we can restrict ourselves to the cases of normal dispersion in both waves ($\sigma_o=1$, $\sigma_m=1$) and anomalous dispersion in the optical wave and normal dispersion in the microwave ($\sigma_o=-1$, $\sigma_m=1$).

MI is found to exist in a large range of system parameters, i.e., the amplitude of the optical CW solution a_o and the velocity mismatch δ. As shown in Fig. 1, a stationary solution can be modulationally unstable in all regimes of dispersion. This is somehow in contrast to MI in Kerr media governed by the nonlinear Schrödinger equation, where MI only occurs when the nonlinearity and the dispersion have opposite signs [1]. In both dispersion domains the MI gain becomes narrowband and increases with the increasing negative velocity mismatch.

In Fig. 2 the boundaries between stable and unstable domains $D=0$ are depicted for both dispersion regions. The bandwidth of unstable frequencies increases with the increasing amplitude of the optical wave. For normal dispersion in both waves CW solutions are generally modulationally unstable. For normal microwave dispersion and anomalous dispersion in the optical wave, a threshold amplitude for the onset of MI exists for positive velocity mismatch $\delta>0$ given by

$$a_{oT}=\sqrt{\frac{4\delta^3}{27}}. \tag{16}$$

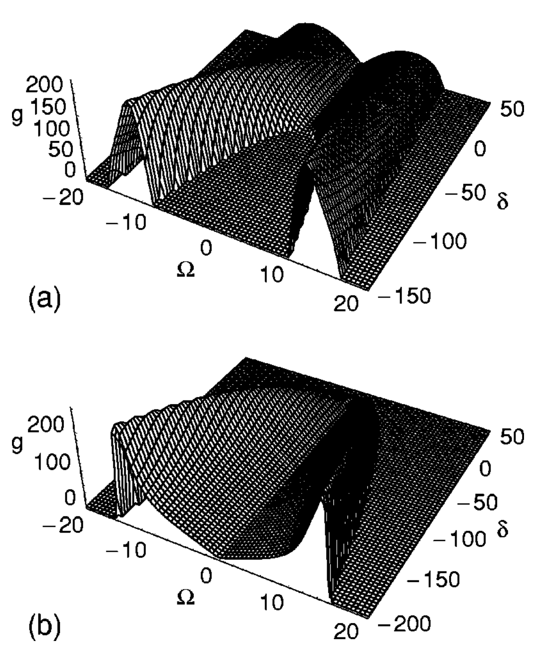

FIG. 1. Gain g versus frequency detuning Ω and the velocity mismatch δ; amplitude of the optical wave $a_o=100$. (a) Normal dispersion for both waves $\sigma_o=1$, $\sigma_m=1$, (b) anomalous dispersion for optical wave $\sigma_o=-1$, normal dispersion for microwave $\sigma_m=1$.

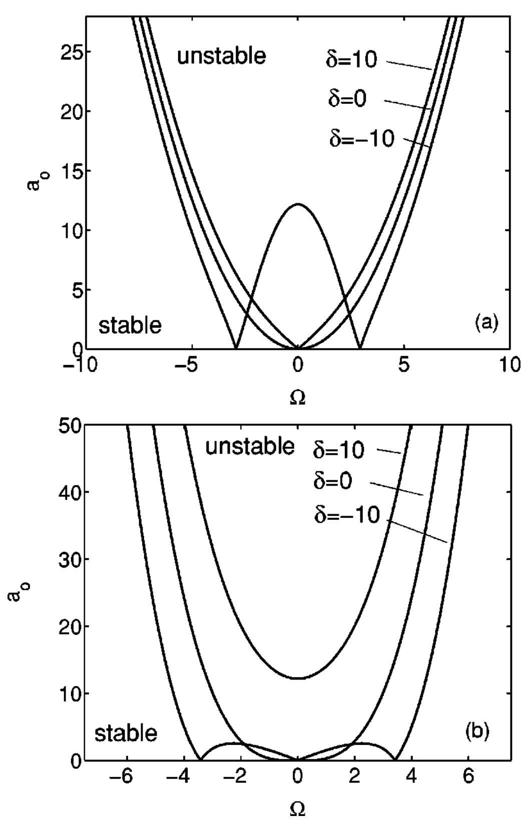

FIG. 2. Boundary between stable and unstable domains $D=0$. (a) Normal dispersion for both waves $\sigma_o=1$, $\sigma_m=1$, (b) anomalous dispersion for optical wave $\sigma_o=-1$, normal dispersion for microwave $\sigma_m=1$.

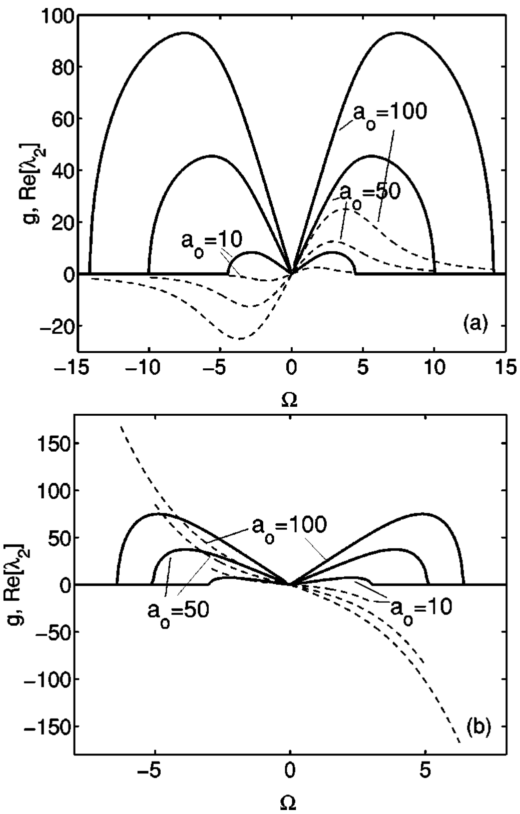

FIG. 3. MI gain g (solid line) vs detuning Ω for different amplitudes a_o of CW solution ($\delta=0$). Also shown is the real part of eigenvalue $\mathrm{Re}[\lambda_2]$ (dashed line). (a) Normal dispersion for both waves $\sigma_o=1$, $\sigma_m=1$, (b) anomalous dispersion for optical wave $\sigma_o=-1$, normal dispersion for microwave $\sigma_m=1$.

Optical signals with amplitudes below this critical value can be operated in electro-optical modulators without risking a spontaneous filamentation.

As shown in Fig. 3, the magnitude of the MI gain grows with the amplitude of the CW solution and the peak gain detuning shifts to larger frequencies. Further, the eigenvalue corresponding to the instability always exhibits a real part. It follows that exponentially growing modulations travel with a velocity different from that of the optical wave.

III. FILAMENTATION AND GENERATION OF ELECTRICAL SIGNALS

MI due to interaction with a microwave field may be observed experimentally by filamentation of high power optical pulses. Here pulses broader than the width corresponding to the sideband frequency with maximum gain $T_{\mathrm{MI}}=2\pi/\Omega_{\max}$ can develop a spontaneous modulation out of noise and finally break up into filaments.

The influence of MI on the propagation of optical pulses was studied by the integration of Eqs. (1) and (2) with a Crank-Nicholson scheme. Here a Gaussian input pulse is perturbed by random noise and subsequently propagated. The initial profiles of the amplitudes $u_{m,o}(z,t)$ are given by

$$u_o(0,t)=A_o\exp[-2\ln(1/2)(t/T_{\mathrm{FWHM}})^2][1+\Delta F_{\mathrm{rand}}(t)], \tag{17}$$

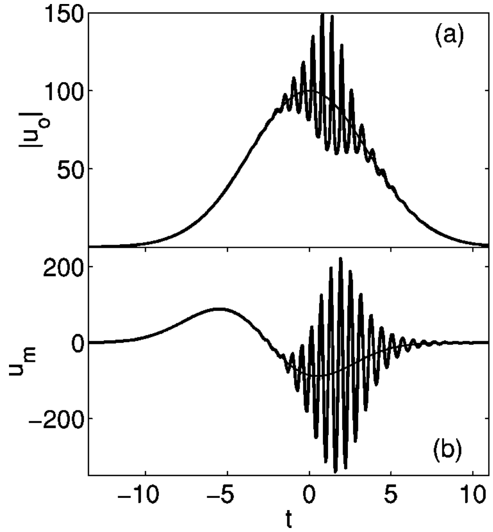

FIG. 4. Filamentation due to modulational instability ($\delta=-100$, $\sigma_o=1$, $\sigma_m=1$, $T_{\mathrm{FWHM}}=6$, $A_o=100$). (a) Optical pulse, thin solid curve; input pulse, thick solid curve, after propagation length $z=0.05$ with $\Delta=10^{-3}$. (b) Microwave signal after propagation length $z=0.05$; thin solid curve, $\Delta=0$; thick solid curve, $\Delta=10^{-3}$.

$$u_m(0,t)=0, \tag{18}$$

where T_{FWHM} is the width and A_o the peak amplitude of the optical input pulse. Δ is the perturbation amplitude with $F_{\mathrm{rand}}(t)$ representing a noise source that gives random numbers from a uniform distribution in the interval $[-1,1]$.

Figure 4 shows the outcome of a corresponding numerical experiment. The propagated perturbed optical pulse develops a modulation due to MI. In this example we added initial noise to the optical pulse with $\Delta=10^{-3}$. However, filamentation can also be observed without an additional noise source ($\Delta=0$) out of numerical noise, albeit at larger propagation lengths.

As depicted in Fig. 4(b), the propagating optical pulse generates a microwave signal by optical rectification. It has been shown in Ref. [9] that, for a propagation length shorter than the walk-off length $L_{\mathrm{wo}}=T_{\mathrm{FWHM}}/\delta$, the microwave evolves as a quasi-single-cycle signal with a midfrequency approximately given by $\Omega=\pi/T_{\mathrm{FWHM}}$, i.e., a frequency determined exclusively by the optical pulse width. However, with increasing initial noise the microwave signal develops an additional high-frequency modulation due to the onset of MI. This is illustrated in Fig. 5, where the frequency spectrum of the generated microwave signal is depicted together with the corresponding gain due to MI. The low-frequency spectrum is determined by the evolving single-cycle microwave signal with a midfrequency that does not significantly change with varying system parameters. The filamentation due to MI manifests itself in the high-frequency spectrum. In contrast to the rectification-induced microwave signal, the midfrequency of the MI-induced modulation is determined by the detuning $\Omega_{\max}$ at peak gain $g_{\max}$ and hence the amplitude of the optical pulse and the velocity mismatch between optical wave and microwave. As shown in Fig. 5, the

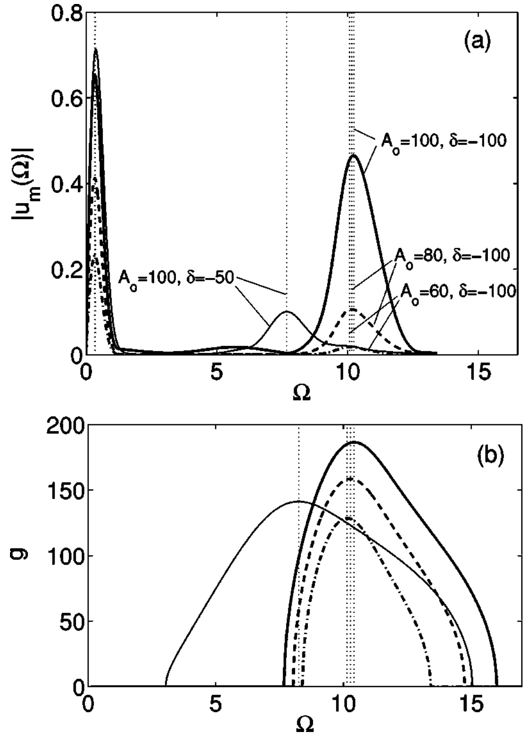

FIG. 5. (a) Frequency spectra of the generated microwave signal for different amplitudes A_o of the injected optical pulse and varying velocity mismatch δ ($\sigma_o=1$, $\sigma_m=1$, $T_{\mathrm{FWHM}}=6$, $z=0.05$, $\Delta=10^{-3}$), the thick solid curve corresponds to the MI-modulated microwave signal in Fig. 4(b), the thin dotted lines mark the peak values of the spectra; (b) corresponding MI gain curves for continuous waves with $a_o=A_o$, the thin dotted lines mark the detuning frequencies $\Omega_{\max}$ at maximum MI gain.

spectral evolution of the high-frequency components of the generated microwave signal is in good agreement with the MI gain curves. This principle could be applied to the generation of narrow-bandwidth ultrashort electrical signals from high power optical pulses. Here, the frequency of the generated microwave signal can be adjusted by tuning the microwave velocity, a principle which is a common practice in the design of traveling wave electro-optic modulators [13,14]. Changing the power of the injected optical pulse allows a further fine tuning of the MI-induced spectral components.

Finally, some comments shall be made about the physical magnitudes involved. In a recent paper [15], typical parameters for a modulator structure consisting of an optical ridge waveguide combined with a coplanar microwave transmission line based on the $\mathrm{Al}_{1-x}\mathrm{Ga}_x\mathrm{As}$ system were estimated. The dimensions of the structure are not reproduced here for brevity. Using the unscaled values in Ref. [15], a scaled optical amplitude $a_o=100$ and a velocity mismatch $\delta=-100$ correspond to an optical power of $P_o=1.8$ kW and an unscaled velocity mismatch of $\Delta n=n_{\mathrm{mic}}-n_{\mathrm{opt}}=-0.034$, where n_{mic} and n_{opt} are the effective group indices of the microwave and the optical wave, respectively. The unscaled maximum MI gain is then given by $g_{\mathrm{us}}=1.9$ cm^{-1} at a frequency $f=1.5$ THz. Keeping in mind that for a spontaneous modulation out of noise the sample length should be a few gain lengths $l_g = 1/g_{\mathrm{us}}$, the gain obtained in this example seems feasible for an experimental observation. However, high microwave losses might shift the net gain to smaller values.

IV. CONCLUSION

In conclusion, we have shown that a strong coupling between an optical wave and a microwave in a dispersive medium with a second order nonlinearity can lead to a modulational instability of optical CW solutions. For normal dispersion in both waves, plane waves are always modulationally unstable. For different signs of the dispersion coefficients a threshold for the onset of MI can exist depending on velocity mismatch and the amplitude of the CW solution. MI and the associated filamentation of high power optical pulses were shown to potentially result in the generation of microwave signals with frequencies considerably higher than the midfrequency of the single-cycle signal obtained by optical rectification.

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